Four Operators – Four Spaces

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Let X be a space of functions $x : [0,1] \to \mathbb{R}$ (to be specified below). In this talk we are going to consider the following four **operators**: The (linear) *multiplication operator* $M_\mu : X \to X$ defined by $(M_\mu x)(t) := \mu(t)x(t)$, where $\mu : [0,1] \to \mathbb{R}$, the (linear) *substitution operator* $\Sigma_\varphi : X \to X$ defined by $(\Sigma_\varphi x)(t) := x(\varphi(t))$, where $\varphi : [0,1] \to [0,1]$, the (nonlinear) *composition operator* $C_f : X \to X$ defined by $(C_f x)(t) := f(x(t))$, where $f : \mathbb{R} \to \mathbb{R}$, and the (nonlinear) *superposition operator* $S_g : X \to X$ defined by $(S_g x)(t) := g(t, x(t))$, where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$.

The four **spaces** we will work in are the Banach space $X = C[0,1]$, equipped with the norm $\|x\|_C := \max\{|x(t)| : 0 \le t \le 1\}$, the Banach space $X = C^1[0,1]$, equipped with the norm $\|x\|_{C^1} := |x(0)| + \|x'\|_C$, the Banach space $X = Lip[0,1]$, equipped with the norm $\|x\|_{Lip} := |x(0)| + lip(x; [0,1])$, and the Banach space $X = BV[0,1]$, equipped with the norm $\|x\|_{BV} := |x(0)| + var(x; [0,1])$.

The **properties** of the four operators $A \in \{M_\mu, \Sigma_\varphi, C_f, S_g\}$ in the four spaces $X \in \{C, C^1, Lip, BV\}$ we are interested in are injectivity, surjectivity, boundedness, continuity, Lipschitz continuity, and compactness. Despite their simple form, these operators exhibit many surprises and pathologies. This is illustrated by a collection of 20 counterexamples which might be useful for your lectures and seminars on real analysis.

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